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The Virasoro group and Lorentzian surfaces: the hyperboloid of one sheet

Christian Duval [a], Laurent Guieu [b,*]

[a] *CPT-CNRS, Luminy Case 907, F-13288 Marseille Cedex 9, France*

[b] *Getodim – Département de Mathématiques, Université de Montpellier II, F-34095 Montpellier Cedex 5, France*

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Abstract

We investigate, in detail, symplectic equivalence between several conformal classes of Lorentz metrics on the hyperboloid of one sheet $H^{1,1} \cong \mathbb{T} \times \mathbb{T} - \Delta$ and affine coadjoint orbits of the group $\mathrm{Diff}_+(\Delta)$ of orientation preserving diffeomorphisms of $\Delta \cong \mathbb{T}$ with its natural projective structure. This will allow for generalizations, namely, to the case of arbitrary projective structures on null infinity.

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1. Introduction

According to the Riemann uniformization theorem, there exists only three conformal types of simply connected Riemannian surfaces, namely

$$\begin{array}{ccc} S^2 & \mathbb{R}^2 & H^2 \\ K=1 & K=0 & K=-1. \end{array}$$

* Corresponding author. E-mail addresses: duval@cpt.univ-mrs.fr (C. Duval), guieu@math.univ-montp2.fr (L. Guieu).

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In the Lorentz case considered in this paper, the relevant geometry is the so-called "fourth" geometry of Poincaré [25] as opportunely mentioned in [19], i.e., the Lorentz geometry of the hyperboloid of one sheet

$$H^{1,1}$$

$$K = \pm 1.$$

"LA QUATRIÈME GÉOMÉTRIE. — Parmi ces axiomes implicites, il en est un qui semble mériter quelque attention, parce qu'en l'abandonnant, on peut construire une quatrième géométrie aussi cohérente que celle d'Euclide, de Lobatchevsky et de Riemann. [...] Je ne citerai qu'un de ces théorèmes et je ne choisirai pas le plus singulier : *une droite réelle peut être perpendiculaire à elle-même*."

Henri Poincaré
La science et l'hypothèse (1902)

Let us, nevertheless, emphasize that a Lorentz uniformization theorem is still not available, as of today — the problem lying in the classification of the conformal boundaries [20,31].

This study has been triggered by previous work of Kostant and Sternberg [18,19] who first pointed out an intriguing relationship between the Schwarzian derivative of a diffeomorphism of null infinity $\mathbb{T}$ of the Lorentz hyperboloid $H^{1,1}$ and the transverse Hessian of the conformal factor associated with this diffeomorphism (viewed as a conformal transformation of $H^{1,1}$). We contend that this correspondence stems from a particular geometric object, namely the cross-ratio as a four-point function associated with the canonical projective structure of the projective line.

Such an observation prompted us to further investigate the relationship between (i) the conformal geometry of the hyperboloid of one sheet $H^{1,1}$ and (ii) the Virasoro group, Vir.

Our contribution has therefore consisted in identifying several conformal classes of Lorentz metrics on $H^{1,1} \cong \mathbb{T}^2 - \Delta$ within the space of projective structures on $\Delta \cong \mathbb{T}$, i.e., the (regular) dual of Vect($\mathbb{T}$) [16]. In doing so, we have been able to give an explicit, yet non-standard, realization of the generic coadjoint orbits [12,13,16,17,33] of the Virasoro group in the framework of two-dimensional real conformal geometry. Note that Iglesias [15] has also obtained other realizations of such orbits in quite a different context.

The paper is organized as follows.

- Section 2 describes in various ways the Lorentz cylinder $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta$ and its associated conformal structure for special projective structures of null infinity, i.e., the circle $\mathcal{S}$.
- In Section 3, we briefly introduce the Schwarzian 1-cocycle S of $\mathrm{Diff}_+(\mathcal{S})$, while in Section 4, we recall the Kostant–Sternberg Theorem [19] and the basic notions attached to conformal Lorentz structures on surfaces.
- Our main results are presented in Section 5 where special, infinite-dimensional, conformal classes of metrics g on $\mathcal{H}$ are shown to be symplectomorphic to coadjoint orbits of the group Vir — central extension of $\mathrm{Conf}_+(\mathcal{H}) \cong \mathrm{Diff}_+(\mathcal{S})$. The $\mathrm{Conf}_+(\mathcal{H})$-orbit of the flat Lorentz metric on the cylinder corresponds to a zero central charge orbit, whereas the central charge c of the other generic Vir-orbits we investigate is related to

the (constant) curvature K of $(\mathcal{H}, \mathrm{g})$ by $cK = 1$. We, likewise, derive the Bott–Thurston cocycle within the same framework.

- Some perspectives are finally drawn in Section 6. It is, in particular, expected that our results allow for generalizations that would, e.g., relate Kulkarni's Lorentz surfaces and universal Teichmüller space.

2. The Lorentz hyperboloid of one sheet

2.1. An adjoint orbit in $\mathfrak{sl}(2, \mathbb{R})$

The single sheeted hyperboloid $H_c^{1,1} \hookrightarrow \mathbb{R}^{2,1}$ defined for $c \in \mathbb{R}_+^*$ by

$$x^2 + y^2 - t^2 = c \tag{2.1}$$

carries a canonical Lorentz metric [1] given by the induced quadratic form

$$\mathrm{g}_c = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2. \tag{2.2}$$

Proposition 2.1.1 [20,34]. *The hyperboloid of one sheet $H_c^{1,1} \cong \mathbb{R} \times \mathbb{T}$ with radius $r = \sqrt{c} \neq 0$ is the homogeneous space*

$$H_c^{1,1} = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1)$$

which is symplectomorphic to the $\mathrm{SL}(2, \mathbb{R})$*-adjoint orbit of*

$$\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}).$$

As a Lorentz manifold, $H_c^{1,1}$ is a space form of constant curvature [2]

$$K = \frac{1}{c} \tag{2.3}$$

whose group of direct isometries is $\mathrm{PSL}(2, \mathbb{R})$.

Remark 2.1.1. *The unit hyperboloid $H_1^{1,1}$ is also symplectomorphic to the manifold of oriented geodesics of the Poincaré disk $H^2 \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$.*

From now on we will write H as a shorthand notation for $H^{1,1}$ provided no confusion occurs.

The following expression for the Lorentz metric (2.2) on H will prove useful. In view of (2.1), write $x = \varrho \sin\theta$, $y = \varrho\cos\theta$, $r = \varrho\sin\phi$, $t = \varrho\cos\phi$ so that the metric (2.2)

[1] In the physics literature $H_1^{1,1}$ is called anti-de Sitter spacetime.

[2] Since $\mathrm{g} \longrightarrow -\mathrm{g}$ yields $K \longrightarrow -K$ and preserves the Lorentz signature $(+, -)$, we will admit $c < 0$ in (2.3); see [10]. Recall that $K = \frac{1}{2}R$, where R is the scalar curvature.

takes the form $g_c = r^2 \csc^2 \phi (\mathrm{d}\theta^2 - \mathrm{d}\phi^2)$. Putting now $\theta_1 = \theta + \phi$ and $\theta_2 = \theta - \phi$, we obtain

$$g_c = \frac{4c\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2}{|\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_2}|^2} \tag{2.4}$$

with (see (2.3))

$$c \in \mathbb{R}^*, \tag{2.5}$$

yielding the canonical Killing metric on the hyperboloid

$$H \cong \mathbb{T} \times \mathbb{T} - \Delta \tag{2.6}$$

globally parametrized by $\theta_1, \theta_2 \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with $\theta_1 \neq \theta_2$. See, e.g., [18]. The transverse null foliations $\theta_1 =$ const. and $\theta_2 =$ const. correspond to the rulings of the hyperboloid, and the diagonal Δ is the conformal boundary [20] (or null infinity [24]) of H.

2.2. The Cayley–Klein model

The material of this section has been borrowed from [4] with a slight adaptation to our framework.

Definition 2.2.1. An *involution* of $\mathbb{R}P^1$ is a homography $s \in \mathrm{PGL}(2, \mathbb{R})$ such that $s^2 = \mathrm{id}$ and $s \neq \mathrm{id}$. We will denote by $\mathcal{I}$ the space of involutions.

In the projective plane P associated to the vector space $\mathfrak{sl}(2, \mathbb{R})$, there is a distinguished conic C, defined by the light cone.

Lemma 2.2.1. *The space of involutions is naturally identified with $P - C$.*

The determinant map $\det : \mathrm{GL}(2, \mathbb{R}) \longrightarrow \mathbb{R}^*$ descends, after projectivization, as a map $\delta : \mathrm{PGL}(2, \mathbb{R}) \longrightarrow \mathbb{Z}/(2\mathbb{Z})$, that defines the two connected components of the projective group. Then, we can define $\mathcal{I}_+ = \mathcal{I} \cap \delta^{-1}(1)$ the space of direct involutions and $\mathcal{I}_- = \mathcal{I} \cap \delta^{-1}(-1)$ the space of anti-involutions. Let us denote by D the interior of the convex hull of C and by $\mathcal{K}$ the complement of $D \cup C$ in P.

Proposition 2.2.1. *The space of direct involutions is naturally isomorphic to the disk D and the space of anti-involutions to $\mathcal{K}$.*

Remark 2.2.1. *Topologically, $\mathcal{K}$ is a Möbius band.*

Proposition 2.2.2. *The twofold covering of orientations for $\mathcal{I}_-$ is $C \times C - \Delta$. The restriction of the projection $\pi : \mathfrak{sl}(2, \mathbb{R}) - \{0\} \longrightarrow P$ to the Lorentz hyperboloid H is a twofold covering on $\mathcal{K}$.*

There exists an isomorphism $P \cong \mathbb{R}P^2$ such that the conic C is mapped onto the unit circle $\mathbb{T}$ in the affine plane $\{t = 1\}$, where x, y, t are homogeneous coordinates in $\mathbb{R}^3$. This isomorphism is given by the map

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \longmapsto \frac{1}{2}\begin{pmatrix} 2a \\ b+c \\ b-c \end{pmatrix}.$$

Thus, we verify that the light cone, whose equation is given by $\det(X) = 0$, is mapped onto the conic of homogeneous equation $x^2 + y^2 - t^2 = 0$.

In the Klein model, the complement $\mathcal{K} = \{z \in \mathbb{C} \mid |z| > 1\}$ of the closed unit disk thus represents the projectivized hyperboloid $P(H)$ in $\mathbb{R}P^2 \cong P(\mathfrak{sl}(2, \mathbb{R}))$. It is the space of geodesics of the open unit disk, i.e., of the hyperbolic plane in the Klein model. See Remark 2.1.1.

2.3. *Projective structures*

In order to gain some insight into the preceding results, let us briefly recall the notion of projective structure [2,3,5,32]. To that end, we need the

Definition 2.3.1. A *projective structure* ϖ on an n-dimensional connected manifold $\mathcal{M}$ is given by the following data:
(1) an immersion $\Phi : \widetilde{\mathcal{M}} \longrightarrow \mathbb{R}P^n$ defined on the universal covering $\widetilde{\mathcal{M}}$ of $\mathcal{M}$,
(2) a homomorphism $T : \pi_1(\mathcal{M}) \longrightarrow \mathrm{PSL}(n+1, \mathbb{R})$
such that

$$\forall a \in \pi_1(\mathcal{M}) \quad \Phi \circ a = T(a) \circ \Phi. \tag{2.7}$$

One calls Φ the *developing map* and T the *holonomy* of the structure.

We denote by $\varpi = [\Phi, T]$ the associated projective structure. The developing map and the holonomy characterizes the structure up to conjugation by the projective group, i.e.,

$$\forall A \in \mathrm{PGL}(n+1, \mathbb{R}) \quad [A \circ \Phi, A \cdot T \cdot A^{-1}] = [\Phi, T].$$

Such a structure is equivalently given by an atlas of projective charts $\varphi_i : U_i \subset \mathcal{M} \longrightarrow \mathbb{R}P^n$ with transition diffeomorphisms in $\mathrm{PGL}(n+1, \mathbb{R})$.

In the one-dimensional case under study, and, more particularly in the case of the circle $\mathcal{S}$, a projective structure ϖ is given by a pair (Φ, M) with $\Phi : \mathbb{R} \longrightarrow \mathbb{R}P^1$ an immersion and $M \in \mathrm{PSL}(2, \mathbb{R})$. Condition 2.7 then reads

$$\Phi(\theta + 2\pi) = M \cdot \Phi(\theta).$$

It is a classic result [13,28] that the space $\mathcal{P}(\mathcal{S})$ of all projective structures on $\mathcal{S}$ is an affine space modeled on the space $\mathcal{Q}(\mathcal{S})$ of quadratic differentials $q = u(\theta)\,\mathrm{d}\theta^2$ of $\mathcal{S}$.

The projective atlas associated with q is obtained by locally solving the third-order non-linear differential equation $q = S(\Phi)$, where S stands for the Schwarzian derivative (see below).

From now on, we restrict considerations to either choices of projective structures on $\mathcal{S}$, namely

(1) the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ defined by the following developing map[3] (with trivial holonomy)

$$\Phi(\theta) = [\mathrm{e}^{\mathrm{i}\theta}] \qquad \text{or} \qquad \Phi(\theta) = 2\tan\frac{\theta}{2}, \tag{2.8}$$

(2) the projective line $\mathbb{R}P^1$ defined by the developing map

$$\Phi(\theta) = \tan\theta \qquad \text{or} \qquad \Phi(t) = t. \tag{2.9}$$

2.4. Lorentzian metric and cross-ratio

2.4.1. First approach

Let us describe, following Ghys [10], how the canonical Lorentz metric (2.4) on anti-de Sitter space (2.6) indeed originates from the cross-ratio

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \tag{2.10}$$

of four points on the projective line [3].

Let us fix $(\theta_1, \theta_2) \in \mathbb{T}^2 - \Delta$ and consider then a nearby point $(\theta_3, \theta_4) = (\theta_1 + \mathrm{d}\theta_1, \theta_2 + \mathrm{d}\theta_2)$. Put $z_j = \mathrm{e}^{\mathrm{i}\theta_j}$ for $j = 1, \ldots, 4$ and perform a Taylor expansion of the cross-ratio (2.10) at (θ_1, θ_2), so that

$$\begin{aligned}(z_1, z_2, z_3, z_4) &= \frac{\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}(1 - \mathrm{e}^{\mathrm{i}\,\mathrm{d}\theta_1})(1 - \mathrm{e}^{\mathrm{i}\,\mathrm{d}\theta_1})}{(\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}(\theta_2 + \mathrm{d}\theta_2)})(\mathrm{e}^{\mathrm{i}\theta_2} - \mathrm{e}^{\mathrm{i}(\theta_1 + \mathrm{d}\theta_1)})} \\ &= \frac{(-i\,\mathrm{d}\theta_1)(-i\,\mathrm{d}\theta_2)}{(\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_2})(\mathrm{e}^{\mathrm{i}\theta_2} - \mathrm{e}^{\mathrm{i}\theta_1})\mathrm{e}^{-\mathrm{i}\theta_1}\mathrm{e}^{-\mathrm{i}\theta_2}} + \cdots \\ &= \frac{-\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{|\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_2}|^2} + \cdots\end{aligned}$$

where the ellipsis "$\cdots$" stands for "terms of order ≥ 3". One can thus claim that, up to higher order terms, the metric (2.4) on the unit hyperboloid H (2.6) is given by $\mathrm{g}_1 = -4\,(z_1, z_2, z_1 + \mathrm{d}z_1, z_2 + \mathrm{d}z_2) + \cdots$ or, equivalently, by

$$\mathrm{g}_1 = -4 \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\,(z_1, z_2, z_1 + \varepsilon\,\mathrm{d}z_1, z_2 + \varepsilon\,\mathrm{d}z_2) \tag{2.11}$$

which is therefore conspicuously $\mathrm{PSL}(2, \mathbb{R})$-invariant.

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[3] We use the notation $[z] = \mathbb{R}z$ for all $z \in \mathbb{C} - \{0\}$.

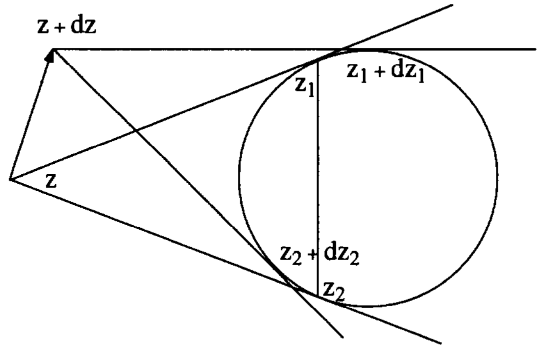

Fig. 1. *The Klein model*

Resorting to Definition 2.3.1, we then have the

Theorem 2.4.1. *Consider the hyperboloid* $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta$ *where the circle* $\mathcal{S}$ *has a projective structure defined by* $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}P^1)$ *as in* (2.8) *or* (2.9). *Then,* $\mathcal{H}$ *carries a natural* $\mathrm{PSL}(2,\mathbb{R})$*-invariant metric of the form*

$$g_1 = (\Phi \times \Phi)^* \frac{4\,\mathrm{d}t_1\,\mathrm{d}t_2}{(t_1 - t_2)^2}. \tag{2.12}$$

Proof. The cross-ratio (2.10) is $\mathrm{PSL}(2,\mathbb{R})$-invariant and so is the Lorentz metric $4\,\mathrm{d}t_1\,\mathrm{d}t_2/(t_1 - t_2)^2$ of $\mathbb{R}P^1 \times \mathbb{R}P^1 - \Delta$ given by (2.11) with $z_j = t_j$ (see (2.9)). In any case (2.8) or (2.9), the metric (2.12) defined on $\mathbb{R}^2 - \Gamma$ where $\Gamma = (\Phi \times \Phi)^{-1}(\Delta)$ is automatically $\pi_1(\mathcal{S})$-invariant thanks to (2.7). It is invariant, as well, under the universal covering $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$. Hence, this metric descends to $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta = \pi \times \pi(\mathbb{R}^2 - \Gamma)$, where $\pi : \mathbb{R} \to \mathcal{S}$ is the universal covering map. The projected metric g_1 is then clearly $\mathrm{PSL}(2,\mathbb{R})$-invariant. □

Example (2.4) corresponds to the developing maps (2.8); as for the first developing map in (2.9), it leads via (2.12) to the metric of the Klein model of Section 2.2 (see Fig. 1).

2.4.2. *Alternative method*

For the sake of completeness, we will again resort to the cross-ratio to derive the canonical Lorentz metric on the space of anti-involutions $\mathcal{I}_-$ introduced in Section 2.2, whence on the unit hyperboloid H. If

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{2.13}$$

is an element of the unit hyperboloid H (that is: $a^2 + bc = 1$), let us consider its image s under the twofold cover

$$p : H \to \mathcal{I}_-$$

given by the anti-involution

$$s(t) = \frac{at+b}{ct-a}. \tag{2.14}$$

Lemma 2.4.1. *The two fixed points of the anti-involution s are*

$$\xi = -\frac{b}{a+1} = \frac{a-1}{c}, \qquad \zeta = -\frac{b}{a-1} = \frac{a+1}{c}.$$

Proof. These are the two real roots of the equation $ct^2 - 2at - b = 0$ expressed with the help of the relation $a^2 + bc = 1$. □

Considering another anti-involution

$$s'(t) = \frac{a't+b'}{c't-a'}$$

prompts us to introduce the cross-ratio (2.10) of the fixed points ξ, ζ, ξ', ζ', namely

$$\beta(s, s') = (\xi, \zeta, \xi', \zeta'). \tag{2.15}$$

Remark 2.4.1. *We suppose that the fixed points are ordered according to the choice of an orientation of the projective line.*

Lemma 2.4.2. *If $\delta X \in T_X H$ is any tangent vector and $\delta s = Tp(\delta X)$, one has*

$$\beta(s, s+\delta s) = -\frac{1}{4c^2}((a\delta c - c\delta a)^2 - (\delta c)^2) + \|(\delta a, \delta c)\|^2 \epsilon(\delta a, \delta c)$$

where $\|\cdot\|$ is the Euclidean norm and $\epsilon(h, k) \to 0$ whenever $(h, k) \to 0$.

Proof. Let us posit $s' = s + \delta s$, i.e., $a' = a + \delta a$, $b' = b + \delta b$ and $c' = c + \delta c$. From definition (2.15) we get

$$\beta(s, s+\delta s) = \left(\frac{a-1}{c}, \frac{a+1}{c}, \frac{a'-1}{c'}, \frac{a'+1}{c'}\right).$$

With the help of two homotheties, we readily find

$$\begin{aligned}\beta(s, s+\delta s) &= (c'(a-1), c'(a+1), c(a'-1), c(a'+1)) \\ &= \frac{(a\delta c - c\delta a)^2 - \delta c^2}{(a\delta c - c\delta a)^2 - (\delta c + 2c)^2}\end{aligned}$$

and the sought formula is obtained by developing this expression up to the second order. □

Theorem 2.4.2. *If g_1 is the (induced) Killing metric on H, one has*

$$\beta(s, s+\delta s) = -\tfrac{1}{4} g_1(\delta X, \delta X) + \|\delta X\|^2 \epsilon(\delta X). \tag{2.16}$$

Proof. Let $X \in H$ be as in (2.13) and $s = p(X)$ be the associated anti-involution (2.14). Recall that the induced Killing metric on H reads

$$g_1(\delta X, \delta X) = -\det(\delta X) = (\delta a)^2 + \delta b \delta c \tag{2.17}$$

for any $\delta X \in T_X H$. With the help of the constraint $a^2 + bc = 1$, we obtain the (local) expressions: $b = (1-a^2)/c$ and $\delta b = -2a\delta a/c + (a^2-1)\delta c/c^2$. Then, Eq. (2.17) becomes

$$g_1(\delta X, \delta X) = \frac{1}{c^2}((a\delta c - c\delta a)^2 - (\delta c)^2)$$

which, together with Lemma 2.4.2, yields the desired result. □

3. The Schwarzian derivative

3.1. Osculating homography of a diffeomorphism

Let $\varphi : \mathbb{R}P^1 \longrightarrow \mathbb{R}P^1$ be a diffeomorphism and let $t_0 \in \mathbb{R}P^1$. We want to find the homography $h \in \mathrm{PGL}(2, \mathbb{R})$ that best approximates the diffeomorphism φ at this point t_0.

Proposition 3.1.1. *This homography h exists and is unique. It is completely defined by the conditions*

$$\begin{aligned} h(t_0) &= \varphi(t_0), \\ h'(t_0) &= \varphi'(t_0), \\ h''(t_0) &= \varphi''(t_0). \end{aligned}$$

The diffeomorphism $h^{-1} \circ \varphi$ has the 2-jet of the identity at t_0. The difference between h and φ starts, hence, at the third-order derivative. (See, e.g., [10].)

Definition 3.1.1. The Schwarzian derivative of φ at the point t_0 is

$$S(\varphi)(t_0) := (h^{-1} \circ \varphi)'''(t_0).$$

The quantity $S(\varphi)(t_0)$ measures how much the diffeomorphism φ differs from a homography at the point t_0. All projective information about φ is encoded into the Schwarzian derivative. If we identify the real projective line with $\mathbb{R} \cup \{\infty\}$ by: $[x, y] \longmapsto t = y/x$, we obtain the classical formula:

$$S(\varphi) = \left(\frac{\varphi'''(t)}{\varphi'(t)} - \frac{3}{2} \frac{\varphi''(t)^2}{\varphi'(t)^2} \right) \mathrm{d}t^2. \tag{3.1}$$

The graph Γ_φ of our diffeomorphism is a simple closed curve on $\mathbb{R}P^1 \times \mathbb{R}P^1$.

Definition 3.1.2. The homography h and its graph Γ_h are, respectively, called the *osculating homography* and the *osculating hyperbola* of φ at t_0.

3.2. The Schwarzian as a projective differential invariant

Theorem 3.2.1 [11]. *The Schwarzian derivative is a third-order complete differential invariant for the group of diffeomorphisms of the projective line.*

More precisely, if φ and ψ are two diffeomorphisms of $\mathbb{R}P^1$, then

$$S(\varphi) = S(\psi) \qquad \Leftrightarrow \qquad \exists A \in \mathrm{PSL}(2,\mathbb{R}),\ \psi = A \circ \varphi.$$

Theorem 3.2.2 [1,16,26,27]. *The Schwarzian S given by* (3.1) *is a non-trivial* 1*-cocycle, i.e.,*

$$S(\varphi \circ \psi) = \psi^* S(\varphi) + S(\psi) \qquad \forall \varphi, \psi \in \mathrm{Diff}_+(\mathbb{R}P^1),$$

on the group of orientation-preserving diffeomorphisms of $\mathbb{R}P^1$ with values in the $\mathrm{Diff}_+(\mathbb{R}P^1)$*-module of real quadratic differentials $\mathcal{Q}(\mathbb{R}P^1)$ of $\mathbb{R}P^1$. Its kernel is* $\mathrm{PSL}(2,\mathbb{R})$.

Remark 3.2.1. *The Schwarzian cocycle* (3.1) *is uniquely characterized (up to a constant factor) by the property of having kernel* $\mathrm{PSL}(2,\mathbb{R})$.

3.3. Cartan formula of the cross-ratio

A useful means for calculating the Schwarzian derivative of an immersion of the projective line is given by

Theorem 3.3.1 [3]. *Consider an immersion $\varphi : \mathbb{R}P^1 \longrightarrow \mathbb{R}P^1$ and four points $t_1, \ldots, t_4 \in \mathbb{R}P^1$ tending to $t \in \mathbb{R}P^1$; putting $\tau_j = \varphi(t_j)$ one has*

$$\frac{(\tau_1, \tau_2, \tau_3, \tau_4)}{(t_1, t_2, t_3, t_4)} - 1 = \frac{1}{6} S(\varphi)(t)(t_1 - t_2)(t_3 - t_4) + [\textit{higher-order terms}], \tag{3.2}$$

where $S(\varphi)$ denotes the Schwarzian derivative (3.1) *of φ.*

This expression still makes sense for any immersion of the circle $\mathcal{S}$ endowed with some projective structure given, e.g., by (2.8) or (2.9). We, indeed, have the

Definition 3.3.1. Let $\varphi : \mathcal{S} \longrightarrow \mathcal{S}$ be an immersion identified with one of its representatives[4] in $C^\infty_{\pi_1(\mathcal{S})}(\mathbb{R})$, then the *Schwarzian* of φ is the pull-back of the Schwarzian (3.2) of the induced map $\tilde{\varphi}$ of $\mathbb{R}P^1$, namely

$$\mathbf{S}(\varphi) = \Phi^* S(\tilde{\varphi}). \tag{3.3}$$

We note that (3.3) yields a well-defined quadratic differential on $\mathcal{S}$ since one trivially finds $a^*\mathbf{S}(\varphi) = \mathbf{S}(\varphi)$ in view of $T(a)^* S(\tilde{\varphi}) = S(\tilde{\varphi})$ for all $a \in \pi_1(\mathcal{S}) \cong \mathbb{Z}$.

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[4] Choose any element of $C^\infty(\mathbb{R})$ that commutes with $\pi_1(\mathcal{S})$.

Proposition 3.3.1. *One has, locally,*

$$\mathbf{S}(\varphi) = S(\varphi) + \varphi^* S(\Phi) - S(\Phi). \tag{3.4}$$

Proof. Using $\tilde{\varphi} \circ \Phi = \Phi \circ \varphi$, one easily finds $\Phi^* S(\tilde{\varphi})(\theta) = S(\varphi)(\theta) + S(\Phi)(\varphi(\theta))\, \varphi'(\theta)^2 - S(\Phi)(\theta)$. □

4. Conformal transformations

4.1. Conformal Lorentz structures

Let us recall some basic definitions and facts about two-dimensional Lorentzian conformal geometry.

Definition 4.1.1 [20]. A *conformal Lorentz structure* on a surface Σ is characterized by a pair of transverse foliations; in other words, it is given by a splitting

$$T\Sigma = T_1\Sigma \oplus T_2\Sigma \tag{4.1}$$

into two trivial line bundles (light-cone field). We call N_1 and N_2, respectively, the spaces of leaves of the two foliations of Σ.

The leaves composing the "grid" associated to these foliations are, locally, given by

$$N_1 : \theta_1 = \text{const.}, \qquad N_2 : \theta_2 = \text{const.}$$

The conformal structure is characterized by the global intersection properties of the (null) leaves of N_1 and N_2.

One can associate to the splitting (4.1) a class of metrics on Σ, locally, of the form $\mathrm{g} = F(\theta_1, \theta_2)\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2$ where F is some smooth positive function. If g is any metric with prescribed null cone field $T_1\Sigma \oplus T_2\Sigma$, we denote by

$$[\mathrm{g}] = \{F \cdot \mathrm{g} | F \in C^\infty(\Sigma, \mathbb{R}_*^+)\} \tag{4.2}$$

the class of metrics conformally equivalent to g. Thus, a conformal Lorentz structure [31] on Σ is equivalently defined by $(\Sigma, [\mathrm{g}])$.

Definition 4.1.2. A diffeomorphism φ of (Σ, g) is called conformal — we write $\varphi \in \mathrm{Conf}(\Sigma, \mathrm{g})$ — if

$$\varphi^*\mathrm{g} = f_\varphi \cdot \mathrm{g} \quad \text{for some } f_\varphi \in C^\infty(\Sigma, \mathbb{R}_*^+). \tag{4.3}$$

The function f_φ is called the *conformal factor* associated with φ.

Remark 4.1.1. *Definition* 4.1.2 *is general and holds in the Riemannian case. It is, for instance, well known that* $\mathrm{Conf}(H^2) = \mathrm{PSL}(2, \mathbb{R})$. *In the Lorentzian case, the conformal*

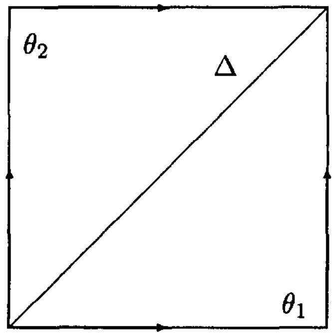

Fig. 2. The hyperboloid

group of $H^{1,1}$ is, however, infinite-dimensional; more precisely, we will see that $\mathrm{Conf}(H^{1,1}) = \mathrm{Diff}(\mathbb{T})$.

4.2. Conformal geometry of the Lorentz hyperboloid

We have seen (2.6) that the global intersection properties of the rulings of the hyperboloid yield (see Fig. 2)

$$H = \mathbb{T} \times \mathbb{T} - \Delta \tag{4.4}$$

whose metric (2.4), (2.11) is given by

$$\mathrm{g}_1 = \frac{4\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{|\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_2}|^2}. \tag{4.5}$$

In view of the previous definitions 4.1.1 and 4.1.2, any conformal (grid-preserving) diffeomorphism φ of a Lorentz surface (Σ, g) is, locally, of the form $\varphi_1 \times \varphi_2$, where $\varphi_j \in \mathrm{Diff}(N_j)$. A (global) conformal diffeomorphism of Σ must preserve the two foliations by lines.

In our case, such a transformation of $\mathbb{T}^2 - \Delta$ must preserve not only the meridians and parallels of $\mathbb{T}^2$, but the diagonal Δ as well. Therefore, $\varphi_1(\theta) = \varphi_2(\theta)$ for all $\theta \in \mathbb{T}$, whence the

Proposition 4.2.1 [19]. *There exists a canonical isomorphism*

$$\mathrm{Diff}(\Delta) \xrightarrow{\cong} \mathrm{Conf}(H)$$

given by the diagonal map: $\varphi \longmapsto \varphi \times \varphi$.

Let us recall the

Theorem 4.2.1 [19].
(i) *Let $\varphi \in \mathrm{Diff}_+(\mathbb{T}) \cong \mathrm{Conf}_+(H)$ be given. Then $f_\varphi = (\varphi^*\mathrm{g}_1)/\mathrm{g}_1 \longrightarrow 1$ as one tends to the conformal boundary Δ.*
(ii) *The conformal factor f_φ extends smoothly to $H \cup \Delta = \mathbb{T}^2$ and has, moreover, Δ as its critical set.*

(iii) *One has* $\mathrm{Hess}(f_\varphi)|\Delta = \frac{1}{3}\widetilde{S}(\varphi)$, *where*

$$\widetilde{S}(\varphi) = S(\varphi) + \tfrac{1}{2}(\varphi'(\theta)^2 - 1)\,\mathrm{d}\theta^2. \tag{4.6}$$

(iv) *The Schwarzian* $\widetilde{S}(\varphi)$ *completely determines* f_φ.

Our proof proceeds as follows. Comparison with the definition (2.11) of the metric g_1 on the hyperboloid H in terms of the cross-ratio prompts the following computation. Given any $\varphi \in \mathrm{Diff}_+(\mathbb{T})$ viewed as a conformal diffeomorphism (4.3) of (H, g_1), apply the Cartan formula (3.2) in the case of a diffeomorphism of the circle $\mathbb{T}$, and get

$$\frac{(\varphi^* g_1)(\theta_1, \theta_2)}{g_1(\theta_1, \theta_2)} - 1 = f_\varphi(\theta_1, \theta_2) - 1, \tag{4.7}$$

$$\frac{(\varphi^* g_1)(\theta_1, \theta_2)}{g_1(\theta_1, \theta_2)} - 1 = \frac{1}{6} S(\tilde{\varphi})(\mathrm{e}^{\mathrm{i}\theta})(\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_2})^2 + \cdots \tag{4.8}$$

where $\tilde{\varphi}(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\varphi(\theta)}$ and $\theta_j \longrightarrow \theta$ for $j = 1, 2$. A tedious calculation using (3.2) leads to

Lemma 4.2.1. *If* $\tilde{\varphi} \in \mathrm{Diff}_+(\mathbb{T})$ *is represented by*[5] $\varphi \in \mathrm{Diff}_{2\pi\mathbb{Z}}(\mathbb{R})$, *one has*

$$S(\tilde{\varphi})(\mathrm{e}^{\mathrm{i}\theta}) = -(S(\varphi)(\theta) + \tfrac{1}{2}(\varphi'(\theta)^2 - 1))\mathrm{e}^{-2\mathrm{i}\theta}. \tag{4.9}$$

From (4.7)–(4.9) one obtains

$$f_\varphi(\theta_1, \theta_2) - 1 = \tfrac{1}{6}(S(\varphi)(\theta) + \tfrac{1}{2}(\varphi'(\theta)^2 - 1))(\theta_1 - \theta_2)^2 + \cdots \tag{4.10}$$

i.e., theorem 1 in [19]. In particular, the conformal factor f_φ extends to the diagonal $\Delta \subset \mathbb{T}^2$ (its critical set) and $f_\varphi|\Delta = 1$, its transverse Hessian being related to the modified Schwarzian derivative (see (4.10)) by $\mathrm{Hess}(f_\varphi) = \frac{1}{3}\widetilde{S}(\varphi)$. The fourth item of Theorem 4.2.1 will be a consequence of Theorem 5.1.2.

We are thus led to the

Theorem 4.2.2.

(i) *Given any* $\varphi \in \mathrm{Conf}_+(H)$ *of* $H = \mathbb{T}^2 - \Delta$ *and* $c \neq 0$, *the twice-symmetric tensor field* $\varphi^* g_c - g_c$ *of* H *extends to null infinity* Δ *and defines a non-trivial* 1*-cocycle*

$$S_c : \varphi \longmapsto \tfrac{3}{2}(\varphi^* g_c - g_c)|\Delta \tag{4.11}$$

of $\mathrm{Diff}_+(\mathbb{T})$ *with values in the module* $\mathcal{Q}(\mathbb{T})$ *of quadratic differentials of the circle, given by the (modified) Schwarzian derivative* (4.6):

$$S_c = c\,\widetilde{S}. \tag{4.12}$$

(ii) *There holds* $H^1(\mathrm{Diff}_+(\mathbb{T}), \mathcal{Q}(\mathbb{T})) = \mathbb{R}[S_1]$.

[5] We denote by $\mathrm{Diff}_{2\pi\mathbb{Z}}(\mathbb{R})$ the universal covering of $\mathrm{Diff}_+(\mathbb{T})$, i.e., the group of those diffeomorphisms φ of $\mathbb{R}$ such that $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$.

Proof. From the formulae (4.10) and (4.5) one immediately gets

$$
\begin{aligned}
(\varphi^* g_1 - g_1)|\Delta &= (\tfrac{2}{3}\,\widetilde{S}(\varphi)(\theta)(\theta_1-\theta_2)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2|\mathrm{e}^{\mathrm{i}\theta_1}-\mathrm{e}^{\mathrm{i}\theta_2}|^{-2}+\cdots)|\Delta\\
&= (\tfrac{2}{3}\,\widetilde{S}(\varphi)(\theta)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2+\cdots)|\Delta\\
&= \tfrac{2}{3}\widetilde{S}(\varphi)(\theta)\,\mathrm{d}\theta^2\\
&= \tfrac{2}{3}\widetilde{S}(\varphi).
\end{aligned}
$$

Then (4.12) is clear by (2.4) and (2.5).

At last, part (ii) follows immediately from the knowledge that $H^1(\mathrm{Diff}_+(\mathbb{T}), \mathcal{Q}(\mathbb{T}))$ is one-dimensional [7,8] and generated by the class of the Schwarzian. □

Remark 4.2.1. *The cocycle* $\varphi \longmapsto \varphi^* g_c - g_c$ *of* $\mathrm{Conf}_+(H)$ *with values in the space of twice-covariant symmetric tensor fields is obviously trivial. Non-triviality of the cocycle* (4.11) *quite remarkably stems from the "restriction" of the latter to null infinity* Δ. [6]

Proposition 4.2.2. *The group of direct isometries of the hyperboloid is*

$$\mathrm{Isom}_+(H, g_c) = \ker(S_c) \cong \mathrm{PSL}(2,\mathbb{R}). \tag{4.13}$$

Proof. Using (4.11), we find that the group $\mathrm{Isom}_+(H, g_c) \subset \mathrm{Diff}_+(\mathbb{T})$ of direct isometries is clearly a subgroup of $\ker(S_c) \cong \mathrm{PSL}(2,\mathbb{R})$. Conversely, for any $\varphi \in \ker(S_c)$, and thanks to (4.8), the conformal factor in (4.7) is $f_\varphi = 1$, i.e., $\varphi \in \mathrm{Isom}_+(H, g_c)$. □

Theorem 4.2.2 still holds true for the $\mathrm{PSL}(2,\mathbb{R})$-invariant metric (2.12) on $\mathcal{S} \times \mathcal{S} - \Delta$. In fact, a calculation akin to that of (4.7), (4.8) leads to

Proposition 4.2.3. *Given any* $\varphi \in \mathrm{Conf}_+(\mathcal{H})$ *of* $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta$, *where* $\mathcal{S}$ *is endowed with the projective structure* (2.8) *or* (2.9), *one has*

$$\mathbf{S}(\varphi) = S_1(\varphi) = \tfrac{3}{2}(\varphi^* g_1 - g_1)|\Delta, \tag{4.14}$$

where the metric g_1 *on* $\mathcal{H}$ *is given by* (2.12) *and the universal Schwarzian* $\mathbf{S}$ *by* (3.3), (3.4).

4.3. Conformal geometry of the flat cylinder

Let us envisage, for a moment, the flat induced Lorentz metric

$$g_0 = \mathrm{d}\theta_1\,\mathrm{d}\theta_2 \tag{4.15}$$

on the cylinder $H = \mathbb{T}^2 - \Delta$. (A non-significant constant factor might be introduced in the definition (4.15) of g_0.)

[6] This observation is due to Valentin Ovsienko.

In this special case, the $\mathrm{Diff}_+(\mathbb{T})$-cocycle S_0 defined, in the same manner as in (4.11), by

$$S_0(\varphi) = (\varphi^* \mathrm{g}_0 - \mathrm{g}_0)|\Delta \tag{4.16}$$

is, plainly, a coboundary since g_0 admits a prolongation to Δ. We, indeed, have $S_0(\varphi)(\theta) = (\varphi'(\theta)^2 - 1)\, \mathrm{d}\theta^2$. Notice that flatness of the metric is now related to triviality of the associated cocycle.

Proposition 4.3.1. *The group of direct isometries of the flat cylinder is*

$$\mathrm{Isom}_+(H, \mathrm{g}_0) = \ker(S_0) \cong \mathbb{T}. \tag{4.17}$$

Proof. Solving $\varphi^* \mathrm{g}_0 = \mathrm{g}_0$ and $\varphi'(\theta) > 0$ gives $\varphi(\theta) = \theta + t$ with $t \in \mathbb{T}$, that is $\varphi \in \ker(S_0)$. $\square$

5. Symplectic structure on conformal classes of metrics on $\mathcal{S} \times \mathcal{S} - \Delta$

We analyze, in this section, the structure of the conformal classes of the previously introduced metrics g_c and g_0 on the "hyperboloid" $\mathcal{H}$ and relate them to the generic coadjoint orbits [16] in the regular dual of the Virasoro group. It should be recalled that the conformal class of g_1 has first been identified with the homogeneous space $\mathrm{Diff}_+(\mathbb{T})/\mathrm{PSL}(2, \mathbb{R})$ in [18].

5.1. Homogeneous space $\mathrm{Diff}_+(\mathcal{S})/\mathrm{PSL}(2, \mathbb{R})$

5.1.1. Conformal classes of curved metrics

Consider first the curved case. If $c \neq 0$, denote by M_c the space of metrics on $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta$ related to $\mathrm{g}_c = c\, \mathrm{g}_1$ (2.4) by a conformal diffeomorphism (see (4.2)), viz.

$$M_c = \{\mathrm{g} \in [\mathrm{g}_1] \,|\, \mathrm{g} = \varphi^* \mathrm{g}_c, \varphi \in \mathrm{Conf}_+(\mathcal{H})\}.$$

These classes M_c of metrics (see Fig. 3) turn out to have a symplectic structure of their own.

Theorem 5.1.1. *If $c \neq 0$, the homogeneous space*

$$\begin{aligned} M_c &= \mathrm{Im}(\varphi \longmapsto \varphi^* \mathrm{g}_c) \\ &\cong \mathrm{Conf}_+(\mathcal{H})/\mathrm{Isom}_+(\mathcal{H}, \mathrm{g}_c) \end{aligned}$$

is endowed with (weak) symplectic structure ω_c which reads

$$\omega_c(\delta_1 \mathrm{g}, \delta_2 \mathrm{g}) = \frac{3}{2} \int_{\Delta} i_{\xi_1} L_{\xi_2} \mathrm{g}, \tag{5.1}$$

where $\delta_j \mathrm{g} = L_{\xi_j} \mathrm{g}$ with $\xi_j \in \mathrm{Vect}(\mathcal{S})$.

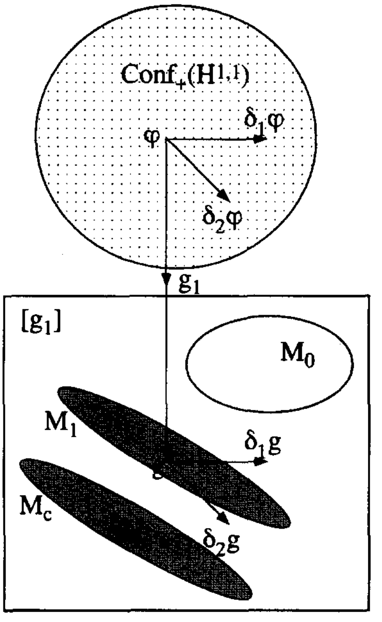

Fig. 3. The conformal classes of metrics on $\mathbb{T}^2 - \Delta$.

Proof. From (5.17) below, ω_c is, indeed, skew-symmetric in its arguments. It is, clearly, also closed. We then have $\delta_2 g_c \in \ker \omega_c$ iff $\omega_c(\delta_1 g_c, \delta_2 g_c) = 0$ for all $\xi_1 \in \mathrm{Vect}(\mathcal{S})$, i.e., iff $L_{\xi_2} g_c|\Delta = 0$, that is iff $\delta_2 g_c = 0$ in view of (4.11) and (4.13). $\square$

We will prove that M_c is symplectomorphic to a Kirillov–Segal–Witten $\mathrm{Diff}_+(\mathcal{S})$-orbit [16,27,33] for the affine coadjoint (anti-)action Coad_Θ on $\mathcal{Q}(\mathcal{S})$ defined by

$$\mathrm{Coad}_\Theta(\varphi)q = \mathrm{Coad}(\varphi)q + \Theta(\varphi), \tag{5.2}$$

where the $\mathrm{Diff}_+(\mathcal{S})$-coadjoint (anti-)action reads

$$\mathrm{Coad}(\varphi)q = \varphi^* q \tag{5.3}$$

and where Θ, a 1-cocycle of $\mathrm{Diff}_+(\mathcal{S})$ with values in $\mathcal{Q}(\mathcal{S})$, is a particular Souriau cocycle [29].

5.1.2. Intermezzo

This technical section presents the standard $\mathrm{Diff}_+(\mathcal{S})$-cocycles in a guise adapted to any projective structure (2.8), (2.9) on the circle $\mathcal{S}$.

Consider the line element

$$\lambda = \Phi^* \mathrm{d}\theta$$

on $\mathcal{S}$ associated with the developing map $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}P^1)$. Actually, λ is a $\pi_1(\mathcal{S})$-invariant line-element of $\mathbb{R}$ which therefore descends to $\mathcal{S}$.

Let φ be a representative in $\mathrm{Diff}_{\pi_1(S)}(\mathbb{R})$ of a diffeomorphism of S and let $\tilde{\varphi} = \Phi \circ \varphi \circ \Phi^{-1}$ denote the diffeomorphism it induces on $\mathbb{R}P^1$.

Proposition 5.1.1.
(i) *The Euclidean cocycle* $\mathbf{E}(\varphi) = \Phi^* E(\tilde{\varphi})$, *where* $E(\tilde{\varphi}) = \log((\tilde{\varphi}^* \,\mathrm{d}\theta)/\,\mathrm{d}\theta)$ *reads*

$$\mathbf{E}(\varphi) = \log\left(\frac{\varphi^*\lambda}{\lambda}\right). \tag{5.4}$$

(ii) *The affine cocycle* $\mathbf{A}(\varphi) = \Phi^* \,\mathrm{d}E(\tilde{\varphi})$ *is then*

$$\mathbf{A}(\varphi) = \mathrm{d}\mathbf{E}(\varphi). \tag{5.5}$$

(iii) *The Schwarzian cocycle* $\mathbf{S}(\varphi) = \Phi^* S(\tilde{\varphi})$ *(see* (3.3), (3.4)*) retains the form*

$$\mathbf{S}(\varphi) = \lambda\, d\left(\frac{\mathbf{A}(\varphi)}{\lambda}\right) - \frac{1}{2}\mathbf{A}(\varphi)^2. \tag{5.6}$$

Proof. We easily prove (iii) by noticing that the Schwarzian (3.1) can be written in terms of the affine coordinate θ of $\mathbb{R}P^1$ as

$$S(\tilde{\varphi}) = \mathrm{d}\theta\, d\left(\frac{\tilde{\varphi}''(\theta)}{\tilde{\varphi}'(\theta)}\right) - \frac{1}{2}\left(\frac{\tilde{\varphi}''(\theta)}{\tilde{\varphi}'(\theta)}\,\mathrm{d}\theta\right)^2$$

and the affine cocycle as $A(\tilde{\varphi}) = (\tilde{\varphi}''(\theta)/\tilde{\varphi}'(\theta))\,\mathrm{d}\theta$. □

For example, the $\mathrm{Diff}_+(\mathbb{T})$-Schwarzian in angular coordinate is recovered with Φ as in (2.8); one finds

$$\mathbf{S}(\varphi)(\theta) = \widetilde{S}(\varphi)(\theta),$$

i.e., the modified Schwarzian derivative (4.6). See also [27].

Proposition 5.1.2. *The infinitesimal Schwarzian takes either forms*

$$\mathbf{s}(\xi) = s_1(\xi)$$

for any $\xi \in \mathrm{Vect}(S)$ *with*[7]

$$\mathbf{s}(\xi) = \lambda\, d\left(\frac{d\,\mathrm{Div}\,\xi}{\lambda}\right) \tag{5.7}$$

and

$$s_1(\xi) = \tfrac{3}{2}(L_\xi g_1)|\Delta.$$

Proof. This follows clearly from (4.14) and (5.6). □

[7] Recall that $\mathrm{Div}\,\xi = (L_\xi \lambda)/\lambda$.

Remark 5.1.1. *In local affine coordinate on* $\mathbb{R}P^1$*, the infinitesimal Schwarzian* (5.7) *of* $\xi = \xi(t)\partial/\partial t$ *retains the familiar form*

$$\mathrm{s}(\xi) = \xi'''(t)\,\mathrm{d}t^2.$$

5.1.3. A Virasoro orbit

With these preparations, let us formulate the

Proposition 5.1.3. *Endow* $\mathrm{Diff}_+(\mathcal{S})$ *with the* 1*-form* α *defined by*

$$\alpha(\delta\varphi) = \frac{1}{2}\int_{\mathcal{S}} \mathbf{A}(\varphi)\delta\mathbf{E}(\varphi), \tag{5.8}$$

where $\delta\varphi = \delta(\varphi\circ\psi)$ *with* $\delta\psi = \xi \in \mathrm{Vect}(\mathcal{S})$ *at* $\psi = \mathrm{id}$.

(i) *The exterior derivative of* α *is given, for* $\xi_1, \xi_2 \in \mathrm{Vect}(\mathcal{S})$*, by*

$$\mathrm{d}\alpha(\delta_1\varphi, \delta_2\varphi) = \int_{\mathcal{S}} \mathbf{S}(\varphi)([\xi_1, \xi_2]) + \underbrace{\int_{\mathcal{S}} \mathrm{d}(\mathrm{Div}\,\xi_1)\,\mathrm{Div}\,\xi_2}_{\mathbf{GF}(\xi_1,\xi_2)}. \tag{5.9}$$

(ii) *If* σ *denotes the canonical symplectic structure of the* $\mathrm{Diff}_+(\mathcal{S})$*-affine coadjoint orbit* $\mathcal{O}$ *of the origin with Souriau cocycle* $\mathbf{S}$ (*see* (5.2)), *namely if*

$$\mathcal{O} = \mathrm{Im}(\mathbf{S}) \tag{5.10}$$

$$\cong \mathrm{Diff}_+(\mathcal{S})/\mathrm{PSL}(2,\mathbb{R}) \tag{5.11}$$

then

$$\mathrm{d}\alpha = \mathbf{S}^*\sigma. \tag{5.12}$$

Proof. Since $\mathrm{d}\alpha(\delta_1\varphi, \delta_2\varphi) = \frac{1}{2}\int_{\mathcal{S}} \mathrm{d}(\delta_1\mathbf{E}(\varphi))\delta_2\mathbf{E}(\varphi) - \frac{1}{2}\int_{\mathcal{S}} \mathrm{d}(\delta_2\mathbf{E}(\varphi))\delta_1\mathbf{E}(\varphi)$ let us first remark that

$$\delta_j\mathbf{E}(\varphi) = \mathbf{A}(\varphi)(\xi_j) + \mathrm{Div}\,\xi_j$$

with the above notation. If we posit for convenience $a = \mathbf{A}/\lambda$, and note that $\lambda(\xi_1)\mathrm{Div}\,\xi_2 - \lambda(\xi_2)\mathrm{Div}\,\xi_1 = \lambda([\xi_1, \xi_2])$, a lengthy calculation then leads to

$$\mathrm{d}\alpha(\delta_1\varphi, \delta_2\varphi) = \int_{\mathcal{S}} (\mathrm{d}a - \tfrac{1}{2}a^2\lambda)\lambda([\xi_1, \xi_2]) + \int_{\mathcal{S}} \mathrm{d}(\mathrm{Div}\,\xi_1)\mathrm{Div}\,\xi_2.$$

Whence the sought Eq. (5.9).

Now, the affine coadjoint orbit of $q_1 \in \mathcal{Q}(\mathcal{S})$ given by the action (5.2) carries a canonical symplectic structure σ which reads [29]:

$$\sigma(\delta_1 q, \delta_2 q) = \langle q, [\xi_1, \xi_2]\rangle + f(\xi_1, \xi_2) \tag{5.13}$$

at $q = \mathrm{Coad}_{\Theta}(\varphi)q_1$; here $f \in Z^2(\mathrm{Vect}(\mathcal{S}), \mathbb{R})$ is the derivative of the group-cocycle $\Theta \in Z^1(\mathrm{Diff}_+(\mathcal{S}), \mathcal{Q}(\mathcal{S}))$ at the identity. The expression (5.9) of $\mathrm{d}\alpha$ clearly matches that of σ (5.13) with $q_1 = 0$, $\Theta = \mathbf{S}$ and $f = \mathbf{GF}$, where the Gelfand–Fuchs cocycle [8] reads

$$\mathbf{GF}(\xi_1, \xi_2) = -\int\limits_{\mathcal{S}} \mathbf{s}(\xi_1)(\xi_2) \tag{5.14}$$

according to (5.7). □

Our main result is then given by

Theorem 5.1.2. *The map*

$$J_c : g \longmapsto \tfrac{3}{2}(\mathrm{g} - \mathrm{g}_c)|\Delta \tag{5.15}$$

establishes a symplectomorphism [8]

$$J_c : (M_c, \omega_c) \longrightarrow (\mathcal{O}_c, \sigma_c) \tag{5.16}$$

between the metrics of $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta$ conformally related to g_c and the affine coadjoint orbit $\mathcal{O}_c = c \cdot \mathcal{O}$ (see (5.11)*) with central charge c, the inverse curvature* (2.3).

Proof. Let us denote by $\mathrm{g}_c : \mathrm{Conf}_+(\mathcal{H}) \longrightarrow M_c$ the orbital map and let us put $\mathrm{g} = \mathrm{g}_c(\varphi) = \varphi^*\mathrm{g}_c$. We find, using (5.1),

$$\begin{aligned}
\omega_1(\delta_1 \mathrm{g}, \delta_2 \mathrm{g}) &= \frac{3}{2}\int\limits_{\Delta} i_{\xi_1} L_{\xi_2}(\mathrm{g} - \mathrm{g}_1) + \frac{3}{2}\int\limits_{\Delta} i_{\xi_1} L_{\xi_2}(\mathrm{g}_1) \\
&= \frac{3}{2}\int\limits_{\Delta} (\mathrm{g} - \mathrm{g}_1)([\xi_1, \xi_2]) + \frac{3}{2}\int\limits_{\Delta} i_{\xi_1} L_{\xi_2}(\mathrm{g}_1) \\
&= \int\limits_{\Delta} S_1(\varphi)([\xi_1, \xi_2]) - \int\limits_{\Delta} s_1(\xi_1)(\xi_2) \\
&= \int\limits_{\Delta} \mathbf{S}(\varphi)([\xi_1, \xi_2]) - \int\limits_{\Delta} \mathbf{s}(\xi_1)(\xi_2)
\end{aligned}$$

with the help of Propositions 4.2.3 and 5.1.2. Note that we have taken into account the skew-symmetry of the Gelfand–Fuchs cocycle introduced in (5.9) and (5.14). One thus gets

$$\omega_1(\delta_1 \mathrm{g}, \delta_2 \mathrm{g}) = \langle \mathbf{S}(\varphi), [\xi_1, \xi_2] \rangle + \mathbf{GF}(\xi_1, \xi_2) \tag{5.17}$$

and, since $\mathrm{g}_c = c\,\mathrm{g}_1$,

$$\omega_c = c\,\omega_1.$$

---

[8] It is the momentum map of the Hamiltonian action of $\mathrm{Conf}_+(\mathcal{H})$ on (M_c, ω_c).

Thanks to (5.9) and (5.12), one can claim that

$$\begin{aligned} \mathrm{d}\alpha &= \mathrm{g}_1^*\omega_1 \\ &= \mathbf{S}^*\sigma. \end{aligned}$$

At last, this clearly entails

$$\omega_c = J_c^*\sigma_c,$$

where $\sigma_c = c\,\sigma$ is the canonical symplectic structure on $\mathcal{O}_c$. □

The following diagram summarizes our claim.

$$\begin{tikzcd} \mathrm{Conf}_+(\mathcal{H}) \arrow[r, "\cong"] \arrow[d, "\mathrm{g}_c"'] & \mathrm{Diff}_+(\mathcal{S}) \arrow[d, "c\,\mathbf{S}"] \\ M_c \arrow[r, "\cong", "J_c"'] & \mathcal{O}_c \end{tikzcd}$$

5.2. Homogeneous space $\mathrm{Diff}_+(\mathcal{S})/\mathbb{T}$

Consider then the flat case (4.15) and introduce the space M_0 of metrics (see Fig. 3) on $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta$ related to g_0 by a conformal diffeomorphism, viz.

$$M_0 = \{\mathrm{g} \in [\mathrm{g}_1] \mid \mathrm{g} = \varphi^*\mathrm{g}_0,\ \varphi \in \mathrm{Conf}_+(\mathcal{H})\}.$$

Theorem 5.2.1. *The homogeneous space*

$$\begin{aligned} M_0 &= \mathrm{Im}(\varphi \longmapsto \varphi^*\mathrm{g}_0) \\ &\cong \mathrm{Conf}_+(\mathcal{H})/\mathrm{Isom}_+(\mathcal{H}, \mathrm{g}_0) \end{aligned}$$

is endowed with a (weak) symplectic structure ω_0 *which reads*

$$\omega_0(\delta_1\mathrm{g}, \delta_2\mathrm{g}) = \int_\Delta i_{\xi_1} L_{\xi_2}\mathrm{g}, \tag{5.18}$$

$$= \int_\Delta \mathrm{g}([\xi_1, \xi_2]), \tag{5.19}$$

where $\delta_j\mathrm{g} = L_{\xi_j}\mathrm{g}$ *with* $\xi_j \in \mathrm{Vect}(\mathcal{S})$.

Proof. Since g_0 can be prolongated to Δ, (5.18) may be rewritten as (5.19) which is manifestly skew-symmetric in its arguments. The closed 2-form ω_0 is weakly non-degenerate as $\delta_2\mathrm{g} \in \ker\omega_0$ iff $L_{\xi_2}\mathrm{g}|\Delta = 0$, i.e., $\delta_2\mathrm{g} = 0$ in view of (4.16) and (4.17). □

In fact, M_0 is symplectomorphic to a $\mathrm{Diff}_+(\mathcal{S})$-coadjoint orbit [16] as shown below.

Let us consider the following quadratic differential

$$q_0 = g_0|\Delta \in \mathcal{Q}(\mathcal{S}) \tag{5.20}$$

so that the $\mathrm{Diff}_+(\mathcal{S})$-coadjoint (anti-)action[9] Coad given by (see (5.3)) $\mathrm{Coad}(\varphi) : q_0 \longmapsto q = \varphi^* q_0$, reads according to (4.16):

$$q = q_0 + S_0(\varphi). \tag{5.21}$$

Proposition 5.2.1. *Endow* $\mathrm{Diff}_+(\mathcal{S})$ *with the* 1*-form* α_0 *defined by*

$$\alpha_0(\delta\varphi) = -\int_{\mathcal{S}} (\varphi^* q_0)(\xi),$$

where, again, $\delta\varphi = \delta(\varphi \circ \psi)$ *with* $\delta\psi = \xi \in \mathrm{Vect}(\mathcal{S})$ *at* $\psi = \mathrm{id}$.

(i) *We have, for any* $\xi_1, \xi_2 \in \mathrm{Vect}(\mathcal{S})$,

$$\mathrm{d}\alpha_0(\delta_1\varphi, \delta_2\varphi) = \int_{\mathcal{S}} (\varphi^* q_0)([\xi_1, \xi_2]).$$

(ii) *The* $\mathrm{Diff}_+(\mathcal{S})$*-coadjoint orbit through* q_0 (5.20) *is*

$$\mathcal{O}_{q_0} = \mathrm{Im}(q_0 \circ \mathrm{Ad}) \tag{5.22}$$

$$\cong \mathrm{Diff}_+(\mathcal{S})/\mathbb{T} \tag{5.23}$$

and is endowed with the symplectic 2*-form* σ_0 *such that*

$$\mathrm{d}\alpha_0 = (q_0 \circ \mathrm{Ad})^* \sigma_0.$$

Proof. If $\delta_j\varphi$ is associated with $\xi_j \in \mathrm{Vect}(\mathcal{S})$ at $\varphi \in \mathrm{Diff}_+(\mathcal{S})$, one readily finds $\delta_j q = L_{\xi_j} q$ and $\mathrm{d}\alpha_0(\delta_1\varphi, \delta_2\varphi) = -\alpha_0([\delta_1, \delta_2]\varphi) = \langle q, [\xi_1, \xi_2]\rangle$ which descends as the canonical symplectic 2-form σ_0 of $\mathcal{O}_{q_0}$, namely

$$\mathrm{d}\alpha_0(\delta_1\varphi, \delta_2\varphi) = \sigma_0(\delta_1 q, \delta_2 q).$$

We then simply check that $\ker(\mathrm{d}\alpha_0)$ is one-dimensional and integrated by $\ker(S_0) \cong \mathbb{T}$ (see (4.17) and (5.21)). □

The "flat" counterpart of Theorem 5.1.2 is now at hand.

Theorem 5.2.2. *The map*

$$J_0 : g \longmapsto g|\Delta \tag{5.24}$$

establishes a symplectomorphism[10]

$$J_0 : (M_0, \omega_0) \longrightarrow (\mathcal{O}_{q_0}, \sigma_0) \tag{5.25}$$

[9] We, indeed, have $\mathrm{Coad}(\varphi)(q_0) = (q_0 \circ \mathrm{Ad})(\varphi)$ for all $\varphi \in \mathrm{Diff}_+(\mathcal{S})$.

[10] It is the momentum map of the Hamiltonian action of $\mathrm{Conf}_+(\mathcal{H})$ on (M_0, ω_0).

between the metrics of $\mathcal{H} = \mathcal{S} \times \mathcal{S} - \Delta$ conformally related to g_0 *and the coadjoint orbit* $\mathcal{O}_{q_0}$ *(see (5.23)) with zero central charge.*

Proof. Clear. □

5.3. Bott–Thurston cocycle and contactomorphisms

It is know since the work of Kirillov [16] that the $\mathrm{Diff}_+(\mathcal{S})$-homogeneous spaces we dealt with in Sections 5.1 and 5.2 are, in fact, genuine coadjoint orbits of the Virasoro group, Vir, i.e., the $(\mathbb{R}, +)$-central extension [30] of $\mathrm{Diff}_+(\mathcal{S})$ that can be recovered as follows in our setting.

Let us emphasize that the 1-form α (5.8) on $\mathrm{Diff}_+(\mathcal{S})$ fails to be invariant. So, let us equip $\mathrm{Diff}_+(\mathcal{S}) \times \mathbb{R}$ with the following "contact" 1-form $\widehat{\alpha}$, viz.

$$\widehat{\alpha}(\delta\varphi, \delta t) = \alpha(\delta\varphi) + \delta t. \tag{5.26}$$

Now, the 2-form $\mathrm{d}\widehat{\alpha}$ is $\mathrm{Diff}_+(\mathcal{S})$-invariant and plainly descends to M_1 as ω_1 (see (5.12) and (5.15), (5.16)). We now have the

Proposition 5.3.1. *Lifting* $\mathrm{Diff}_+(\mathcal{S})$ *into the group of automorphisms of* $(\mathrm{Diff}_+(\mathcal{S}) \times \mathbb{R}, \widehat{\alpha})$ *yields the Virasoro group* Vir *with multiplication law*

$$(\varphi_1, t_1) \cdot (\varphi_2, t_2) = \left(\varphi_1 \circ \varphi_2, t_1 + t_2 \underbrace{-\frac{1}{2} \int_{\mathcal{S}} \mathbf{E}(\varphi_1 \circ \varphi_2) \mathbf{A}(\varphi_2)}_{\mathbf{BT}(\varphi_1, \varphi_2)} \right), \tag{5.27}$$

where $\mathbf{BT}$ *is the Bott–Thurston cocycle* [1] *of* $\mathrm{Diff}_+(\mathcal{S}) \cong \mathrm{Conf}_+(\mathcal{H})$.

Proof. Using the cocycle relation $\mathbf{E}(\varphi \circ \psi) = \psi^* \mathbf{E}(\varphi) + \mathbf{E}(\psi)$ — see (5.4) — and (5.5), (5.8), one immediately finds

$$\begin{aligned} \alpha(\delta(\varphi \circ \psi)) &= \frac{1}{2} \int_{\mathcal{S}} \psi^*(\mathbf{A}(\varphi) \delta(\mathbf{E}(\varphi))) + \frac{1}{2} \int_{\mathcal{S}} \mathbf{A}(\psi) \delta(\mathbf{E}(\varphi \circ \psi)) \\ &= \alpha(\delta\varphi) + \delta \left[\frac{1}{2} \int_{\mathcal{S}} \mathbf{E}(\varphi \circ \psi) \mathbf{A}(\psi) \right] \end{aligned}$$

for all $\varphi, \psi \in \mathrm{Diff}_+(\mathcal{S})$. Looking for those maps $(\varphi, t) \longmapsto (\varphi^\star, t^\star)$ such that $\varphi^\star = \varphi \circ \psi$ and $\widehat{\alpha}(\delta\varphi^\star, \delta t^\star) = \widehat{\alpha}(\delta\varphi, \delta t)$ leads to $t^\star = t + \mathbf{BT}(\varphi, \psi) + \text{const.}$, hence, to the group law (5.27). □

The triple (**S**, **GF**, **BT**) is a special instance of a general structure that has been coined "trilogy of the moment" [15].

Remark 5.3.1. *It would be interesting to have a conformal interpretation of the contact structure* $\mathrm{Vir}/(\ker\widehat{\alpha}\cap\ker\,\mathrm{d}\widehat{\alpha})$ *above* (M_1, ω_1).

6. Conclusion and outlook

This work prompts a series of more or less ambitious questions connected with the striking analogies between conformal geometry of Lorentz surfaces and projective geometry of conformal infinity that we have just discussed. It constitutes an introduction to a more detailed paper (in preparation) where the authors wish to tackle the following problems.

(i) Is it possible to realize any Virasoro coadjoint orbit [11] as a conformal class of Lorentz metrics on the cylinder? If this is so, spell out the symplectic forms in terms of the classes of metrics; also study the relationship between the properties of an orbit and the dynamics of the null foliations in the associated conformal class. There exists, in fact, a map sending the space of Virasoro orbits — modules of projective structures on the circle — to the space of modules of Lorentzian conformal structures on the cylinder; analyze its properties. More conceptually, given a conic C in the real projective plane, what are the links between the space of projective structures on C, the space of Lorentzian structures in the exterior of C and the space of Riemannian metrics in the interior of C?

(ii) The Ghys theorem [9,23] states that any diffeomorphism of the projective line has at least four points where its Schwarzian vanishes, i.e., four points where the contact of the graph of the diffeomorphism with its osculating hyperbola is greater than the generic one. This result is a Lorentzian analogue of the so-called four vertices theorem [12] for closed curves in the Euclidean plane. In our context, the Ghys theorem would imply the existence, for any conformal automorphism of the hyperboloid, of some particular points where this diffeomorphism is closer than usual to an isometry.

(iii) The orbit $\mathrm{Diff}(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ embeds symplectically in the universal Teichmüller space $T(1) = \mathrm{QS}(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$, where $\mathrm{QS}(\mathbb{T})$ denotes the group of quasi-symmetric homeomorphisms of the circle [22]. With the help of the quantum differential calculus of Connes, it is possible to construct extensions of the three fundamental cocycles **E**, **A** and **S** to the group $\mathrm{QS}(\mathbb{T})$ [21]. Can one construct a "quantum analogue" of the Lorentzian hyperboloid whose conformal class may be identified with $T(1)$?

Let us finally mention two other subjects closely connected with our problem, namely the geometry of the Wess–Zumino–Witten model [6] and Douglas' proof of the Plateau problem revisited by Guillemin et al. [14].

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[11] Other isotropy groups are, e.g., the finite coverings of $\mathrm{PSL}(2,\mathbb{R})$ and one-parameter subgroups of the form $\mathbb{T}\times\mathbb{Z}_n$; see [13].

[12] Any closed simple curve in the plane admits at least four points where its Euclidean curvature is critical.

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